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GENERALIZED DESCENT ALGEBRAS

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Abstract. If A is a subset of the set of reflections of a finite Coxeter group W, we define a sub-$\mathbb{Z}$-module $\mathcal{D}_A(W)$ of the group algebra $\mathbb{Z}W$. We provide examples where this submodule is a subalgebra. This family of subalgebras includes strictly the Solomon descent algebra and, if W if of type B, the Mantaci-Reutenauer algebra.

Introduction

Let (W,S) be a finite Coxeter system whose length function is denoted by $\ell : W \to \mathbb{N} = \{0,1,2,\dots\}$. In 1976, Solomon introduced a remarkable subalgebra ΣW of the group algebra $\mathbb{Z}W$, called the *Solomon descent algebra* [8]. Let us recall its definition. If $I \subset S$, let W_I denote the *standard parabolic subgroup* generated by I. Then

$$X_I = \{w \in W \mid \forall\, s \in I,\ \ell(ws) > \ell(w)\}$$

is a set of *minimal length coset representatives* of W/W_I. Let $x_I = \sum_{w\in X_I} w \in \mathbb{Z}W$. Then ΣW is defined as the sub-$\mathbb{Z}$-module of $\mathbb{Z}W$ spanned by $(x_I)_{I\subset S}$. Moreover, ΣW is endowed with a $\mathbb{Z}$-linear map $\theta : \Sigma W \to \mathbb{Z}\operatorname{Irr} W$ satisfying $\theta(x_I) = \operatorname{Ind}_{W_I}^{W} 1_{W_I}$. This is an algebra homomorphism.

If W is the symmetric group $\mathfrak{S}_n$, θ becomes an epimorphism and the pair $(\Sigma\mathfrak{S}_n, \theta)$ provides a nice construction of $\operatorname{Irr}(\mathfrak{S}_n)$ [6], which is the first ingredient of several recent works, see for instance [9, 3]. However, the morphism θ is surjective if and only if W is a product of symmetric groups.

In [4], we have constructed a subalgebra $\Sigma'(W_n)$ of the group algebra $\mathbb{Z}W_n$ of the Coxeter group W_n of type B_n: it turns out that this subalgebra contains ΣW_n, that it is also endowed with an algebra homomorphism $\theta' : \Sigma'(W_n) \to \mathbb{Z}\operatorname{Irr} W_n$ extending θ. Moreover, θ' is surjective and $\mathbb{Q}\otimes_{\mathbb{Z}} \operatorname{Ker}\theta'$ is the radical of $\mathbb{Q}\otimes_{\mathbb{Z}} \Sigma'(W_n)$. This lead to a construction of the irreducible characters of W_n following Jőllenbeck's strategy. In fact, $\Sigma'(W_n)$ is the *Mantaci-Reutenauer algebra* [7].

It is a natural question to ask whether this kind of construction can be generalized to other groups. However, the situation seems to be much more complicated in the other types. Let us explain now what kind of subalgebras we are looking for.

Let $T = \{wsw^{-1} \mid w \in W \text{ and } s \in S\}$ be the set of reflections in W. Let A be a fixed subset of T. If $I \subset A$, we still denote by W_I the subgroup of W generated by I and we still set

$$X_I = \{w \in W \mid \forall\, x \in W_I,\ \ell(wx) \geq \ell(w)\}.$$

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Then X_I is again a set of representatives for W/W_I. Now, let $x_I = \sum_{w \in X_I} w \in \mathbb{Z}W$. Then

$$\Sigma_A(W) = \sum_{I \subset A} \mathbb{Z}x_I$$

is a sub-$\mathbb{Z}$-module of $\mathbb{Z}W$. However, it is not in general a subalgebra of $\mathbb{Z}W$.

Let us define another sub-$\mathbb{Z}$-module of $\mathbb{Z}W$. If $w \in W$, let

$$D_A(w) = \{s \in A \mid \ell(ws) < \ell(w)\} \quad \subset A$$

be the A-descent set of w. A subset I of A is said to be A-*admissible* if there exists $w \in W$ such that $D_A(w) = I$. Let $\mathcal{P}_{\mathrm{ad}}(A)$ denote the set of A-admissible subsets of A. If $I \in \mathcal{P}_{\mathrm{ad}}(A)$, we set $D_I^A = \{w \in W \mid D_A(w) = I\}$ and $d_I^A = \sum_{w \in D_I^A} w \in \mathbb{Z}W$. Now, let

$$\mathcal{D}_A(W) = \bigoplus_{I \in \mathcal{P}_{\mathrm{ad}}(A)} \mathbb{Z}d_I^A.$$

As an example, $\mathcal{D}_S(W) = \Sigma_S(W) = \Sigma W$. The main theorem of this paper is the following (here, $C(w)$ denotes the conjugacy class of w in W):

Theorem A. *If there exists two subsets S_1 and S_2 of S such that $A = S_1 \cup (\cup_{s \in S_2} C(s))$, then $\mathcal{D}_A(W)$ is a subalgebra of $\mathbb{Z}W$.*

This theorem contains the case of Solomon descent algebra (take $A = S$), the Mantaci-Reutenauer algebra $\Sigma'(W_n)$ (take $A = \{s_1, \ldots, s_{n-1}\} \cup C(t)$, where $S = \{t, s_1, \ldots, s_{n-1}\}$ satisfies $C(t) \cap \{s_1, \ldots, s_{n-1}\} = \varnothing$), and one of the subalgebras we have constructed in type G_2. But it also gives a new algebra in type F_4 of $\mathbb{Z}$-rank 300 (take $A = \{s_1, s_2\} \cup C(s_3)$, where $S = \{s_1, s_2, s_3, s_4\}$ satisfies $\{s_1, s_2\} \cap C(s_3) = \varnothing$). Moreover, if $A = T$, we get that $\mathcal{D}_A(W) = \mathbb{Z}W$ (see Example 1.7). In the case of dihedral groups, we get another family of algebras:

Theorem B. *If W is a dihedral group of order $4m$ ($m \geq 1$), $S = \{s, t\}$ and $A = \{s, t, sts\}$ or $A = \{t, sts\}$, then $\mathcal{D}_A(W)$ is a subalgebra of $\mathbb{Z}W$.*

It must be noted that the algebras constructed in Theorems A and B are not necessarily unitary. More precisely $1 \in \mathcal{D}_A(W)$ if and only if $S \subset A$. Moreover, if $S \subset A$, then $\Sigma W \subset \mathcal{D}_A(W)$.

The bad point in the above construction is that, in many of the above cases, one has $\Sigma_A(W) \neq \mathcal{D}_A(W)$. Also, we are not able to construct in general a morphism of algebras $\theta_A : \mathcal{D}_A(W) \to \mathbb{Z}\,\mathrm{Irr}\,W$ extending θ if $S \subset A$. It can be proved that, in some cases, this morphism does not exist.

This paper is organized as follows. Section 1 is essentially devoted to the proofs of Theorems A and B. One of the key step is that the subsets D_I^A are left-connected (recall that a subset E of W is said to be *left-connected* if, for all w, $w' \in E$, there exists a sequence $w = w_1$, w_2,..., $w_r = w'$ of elements of E such that $w_{i+1}w_i^{-1} \in S$ for every $i \in \{1, 2, \ldots, r-1\}$). In Section 2, we discuss more precisely the case of dihedral groups.

1. Descent sets

Let (W, S) be a finitely generated Coxeter system (not necessary finite). If s, $s' \in S$, we denote by $m(s, s')$ the order of $ss' \in W$. If W is finite, we denote by w_0 its longest element.

1.1. **Root system.** Let V be an $\mathbb{R}$-vector space endowed with a basis indexed by S denoted by $\Delta = \{\alpha_s \,|\, s \in S\}$. Let $B : V \times V \to \mathbb{R}$ be the symmetric bilinear form such that

$$B(\alpha_s, \alpha_{s'}) = -\cos\left(\frac{\pi}{m(s,s')}\right)$$

for all s, $s' \in S$. If $s \in S$ and $v \in V$, we set

$$s(v) = v - 2B(\alpha_s, v)\alpha_s.$$

Thus s acts as the reflection in the hyperplane orthogonal to α_s (for the bilinear form B). This extends to an action of W on V as a group generated by reflections. It stabilizes B.

We recall some basic terminology on root systems. The *root system* of (W, S) is the set $\Phi = \{w(\alpha_s) \,|\, w \in W, s \in S\}$ and the elements of Δ are the *simple roots*. The roots contained in

$$\Phi^+ = \big(\sum_{\alpha\in\Delta} \mathbb{R}^+\alpha\big) \cap \Phi$$

are said to be *positive*, while those contained in $\Phi^- = -\Phi^+$ are said to be *negative*. Moreover, Φ is the disjoint union of Φ^+ and Φ^-. If $w \in W$, $\ell(w) = |N(w)|$, where

$$N(w) = \big\{\alpha \in \Phi^+ \,|\; w(\alpha) \in \Phi^-\big\}.$$

Let $\alpha = w(\alpha_s) \in \Phi$, then $s_\alpha = wsw^{-1}$ acts as the reflection in the hyperplane orthogonal to α and $s_\alpha = s_{-\alpha}$. Therefore, the *set of reflections of* W

$$T = \bigcup_{w\in W} wSw^{-1}$$

is in bijection with Φ^+ (and thus Φ^-).

Let us recall the following well-known result:

Lemma 1.1. *Let $w \in W$. Then:*

(a) *If $\alpha \in \Phi^+$, then $\ell(ws_\alpha) > \ell(w)$ if and only if $w(\alpha) \in \Phi^+$.*
(b) *If $s \in S$, then*

$$N(sw) = \begin{cases} N(w) \coprod \{w^{-1}(\alpha_s)\} & \text{if } \ell(sw) > \ell(w), \\ N(w) \setminus \{-w^{-1}(\alpha_s)\} & \text{otherwise.} \end{cases}$$

From now on, and until the end of this paper, we fix a subset A of T. We start with easy observations.

As a consequence of Lemma 1.1 (a), we get that

$$D_A(w) = \big\{s_\alpha \in A \,|\, \alpha \in \Phi^+ \text{ and } w(\alpha) \in \Phi^-\big\}.$$

We also set

$$N_A(w) = \{\alpha \in \Phi^+ \mid s_\alpha \in A \text{ and } w(\alpha) \in \Phi^-\}.$$

The map $N_A(w) \to D_A(w)$, $\alpha \mapsto s_\alpha$ is then a bijection.

1.2. **Properties of the map D_A.** First, using Lemma 1.1 (b), we get:

Corollary 1.2. *If $s \in S$ and if $w \in W$ is such that $w^{-1}sw = s_{w^{-1}(\alpha_s)} \notin A$, then $N_A(w) = N_A(sw)$ (and $D_A(w) = D_A(sw)$).*

Remark 1.3 - If $A_1 \subset A_2 \subset T$, then $D_{A_1}(w) = D_{A_2}(w) \cap A_1$ for all $w \in W$. Therefore if W is finite, $\mathcal{D}_{A_1}(W) \subset \mathcal{D}_{A_2}(W)$. $\square$

Proposition 1.4. *We have:*

(a) *$\varnothing$ is A-admissible.*
(b) *$D_\varnothing^A = \{1\}$ if and only if $S \subset A$.*

Proof. We have $D_A(1) = \varnothing$ so (a) follows. If $s \in S \setminus A$, then $D_A(s) = \varnothing$. This shows (c). $\square$

The notion of A-descent set is obviously compatible with direct products:

Proposition 1.5. *Assume that $W = W_1 \times W_2$ where W_1 and W_2 are standard parabolic subgroups of W. Then, for all $I \in \mathcal{P}_{\mathrm{ad}}(A)$, we have*

$$D_I^A = D_{I\cap W_1}^{A\cap W_1} \times D_{I\cap W_2}^{A\cap W_2}.$$

Corollary 1.6. *Assume that W is finite and that $W = W_1 \times W_2$ where W_1 and W_2 are standard parabolic subgroups of W. Then*

$$\mathcal{D}_A(W) = \mathcal{D}_{A\cap W_1}(W_1) \otimes_{\mathbb{Z}} \mathcal{D}_{A\cap W_2}(W_2).$$

Example 1.7 - Consider the case where $A = T$ (then $N_A(w) = N(w)$). It is well-known [5, Chapter VI, Exercise 16] that the map $w \mapsto N(w)$ from W onto the set of subsets of Φ^+ is injective (observe that if $\alpha \in N(w_1 w_2^{-1})$ then $\pm w_2^{-1}(\alpha)$ lives in the union, but not in the intersection, of $N(w_1)$ and $N(w_2)$). Therefore, the map $W \to \mathcal{P}_{\mathrm{ad}}(T)$, $w \mapsto D_T(w)$ is injective. In particular, if W is finite, then $\mathcal{D}_T(W) = \mathbb{Z}W$. $\square$

In the case of finite Coxeter groups, the multiplication on the left by the longest element has the following easy property.

Proposition 1.8. *If W is finite and if $w \in W$, then $D_A(w_0 w) = A \setminus D_A(w)$.*

Corollary 1.9. *If W is finite, then:*

(a) *A is A-admissible;*
(b) *$I \in \mathcal{P}_{\mathrm{ad}}(A)$ if and only if $A \setminus I \in \mathcal{P}_{\mathrm{ad}}(A)$;*
(c) *$D_A^A = \{w_0\}$ if and only if $S \subset A$.*

Proof. $D_A(w_0) = A$ so (a) follows. (b) follows from Proposition 1.8. (c) follows from Proposition 1.8 and Proposition 1.4 (b). $\square$

1.3. **Left-connectedness.** In [1], Atkinson gave a new proof of Solomon's result by using an equivalence relation to describe descent sets. We extend his result to A-descent sets. It shows in particular that the subsets D_I^A are left-connected.

Let w and w' be two elements of W. We say that w is an *A-descent neighborhood* of w', and we write $w \smile_A w'$, if $w'w^{-1} \in S$ and $w^{-1}w' \notin A$. It is easily seen that $\smile_A$ is a symmetric relation. The reflexive and transitive closure of the A-descent neighborhood relation is called *the A-descent equivalence*, and is denoted by $\sim_A$. The next proposition characterizes this equivalence relation in terms of A-descent sets.

Proposition 1.10. *Let w, $w' \in W$. Then*

$$w \sim_A w' \Leftrightarrow D_A(w) = D_A(w') \Leftrightarrow N_A(w) = N_A(w').$$

Proof. The second equivalence is clear. If $w \smile_A w'$, then it follows from Corollary 1.2 that $N_A(w) = N_A(w')$. It remains to show that, if $N_A(w) = N_A(w')$, then $w \sim_A w'$.

So, assume that $N_A(w) = N_A(w')$. Write $x = w'w^{-1}$ and let $m = \ell(x)$. If $\ell(x) = 0$, then $w = w'$ and we are done. Assume that $m \geq 1$ and write $x = s_1 s_2 \dots s_m$ with $s_i \in S$. We now want to prove on induction on m that

$$(*) \qquad w \smile_A s_m w \smile_A s_{m-1} s_m w \smile_A \cdots \smile_A s_2 \dots s_m w \smile_A s_1 s_2 \dots s_m w = w'.$$

First, assume that $w \not\smile_A s_m w$. In other words, $w^{-1} s_m w \in A$. For simplification, let $\alpha_i = \alpha_{s_i}$. By Lemma 1.1 (b), we have

$$N_A(s_m w) = N_A(w) \coprod \{w^{-1}(\alpha_m)\} \quad \text{or} \quad N_A(w) \setminus \{-w^{-1}(\alpha_m)\}.$$

In the first case, as $N_A(w) = N_A(w')$, and by applying again Lemma 1.1 (b), there exists a step $i \in \{1, 2, \dots, m-1\}$ between $N_A(w)$ to $N_A(w')$ where $w^{-1}(\alpha_m)$ is removed from $N_A(s_i \dots s_m w)$, that is, $(s_{i+1} \dots s_m w)^{-1}(\alpha_i) = -w^{-1}(\alpha_m)$. In the same way, we get the same result in the second case. In other words, we have proved that there exists $i \in \{1, 2, \dots, m-1\}$ such that $s_m \dots s_{i+1}(\alpha_i) = -\alpha_m$, so, by Lemma 1.1 (a), we have $\ell(s_m \dots s_{i+1} s_i) < \ell(s_m \dots s_{i+1})$. This contradicts the fact that $m = \ell(x)$. So $w \smile_A s_m w$, and then $N_A(w) = N_A(s_m w)$. Hence $N_A(s_m w) = N_A(w')$ and $w'(s_m w)^{-1} = s_1 s_2 \dots s_{m-1}$. We get then by induction that $s_m w \smile_A s_{m-1} s_m w \smile_A \cdots \smile_A s_1 s_2 \dots s_m w = w'$, which shows $(*)$. $\square$

Corollary 1.11. *If $I \in \mathcal{P}_{\mathrm{ad}}(A)$, then D_I^A is left-connected.*

1.4. **Nice subsets of T.** We say that A is *nice* if, for every $s \in A$ and $w \in W$ such that $w^{-1} s w \notin A$, we have $D_A(sw) = D_A(w)$. Notice that every subset of S is nice, by Corollary 1.2.

If $w \in W$ and I, $J \in \mathcal{P}_{\mathrm{ad}}(A)$, we set

$$D_A(I, J, w) = \big\{(u, v) \in D_I^A \times D_J^A \mid uv = w\big\}.$$

The next lemma gives an obvious characterization of the fact that $\mathcal{D}_A(W)$ is an algebra in terms of these sets.

Lemma 1.12. *Assume that W is finite. Then the following are equivalent:*

(1) *$\mathcal{D}_A(W)$ is a subalgebra of $\mathbb{Z}W$.*
(2) *For all I, $J \in \mathcal{P}_{\mathrm{ad}}(A)$ and for all w, $w' \in W$ such that $D_A(w) = D_A(w')$, we have $|D_A(I, J, w)| = |D_A(I, J, w')|$.*

If these conditions are fulfilled, we choose for any $I \in \mathcal{P}_{\mathrm{ad}}(A)$ an element z_I in D_I^A. Then

$$d_I^A d_J^A = \sum_{K \in \mathcal{P}_{\mathrm{ad}}(A)} |D_A(I, J, z_K)| d_K^A.$$

Let us fix now $s \in S$ and let $(u, v) \in W \times W$. If $u \smile_A su$, we set $\psi_s^A(u, v) = (su, v)$. If $u \not\smile_A su$, then we set $\psi_s^A(u, v) = (u, u^{-1} suv)$. Note that, in the last case, $u^{-1} su \in A$. We have $(\psi_s^A)^2 = \mathrm{Id}_{W \times W}$. In particular, ψ_s^A is a bijection. Using ψ_s^A, one can relate the notion of nice subsets to the property (2) stated in Lemma 1.12.

Proposition 1.13. *Assume that A is nice. Let $I,\ J \in \mathcal{P}_{\mathrm{ad}}(W)$, let $w \in W$ and let $s \in S$ be such that $w \smile_A sw$. Then $\psi_s^A(D_A(I,J,w)) = D_A(I,J,sw)$.*

Proof. Let (u,v) be an element of $D_A(I,J,w)$. By symmetry, we only need to prove that $\psi_s^A(u,v) \in D_A(I,J,sw)$. If $u \smile_A su$, then $D_A(su) = D_A(u) = I$ by Proposition 1.10, so $\psi_s^A(u,v) = (su,v) \in D_A(I,J,sw)$. So, we may, and we will, assume that $u \not\smile_A su$. Let $s' = u^{-1}su \in A$. Note that $w^{-1}sw = v^{-1}s'v \notin A$. Then, $\psi_s^A(u,v) = (u, s'v)$ and $us'v = sw$. So, we only need to prove that $D_A(s'v) = D_A(v)$. But this just follows from the definition of nice subset of T. $\square$

Corollary 1.14. *If W is finite and if A is nice, then $\mathcal{D}_A(W)$ is a subalgebra of $\mathbb{Z}W$. It is unitary if and only if $S \subset A$.*

Proof. This follows from Lemma 1.12 and from Propositions 1.10 and 1.13. $\square$

Remark 1.15 - It would be interesting to know if the converse of Corollary 1.14 is true. $\square$

1.5. **Proof of Theorems A and B.** Using Corollary 1.14, we see that Theorems A and B are direct consequences of the following theorem (which holds also for infinite Coxeter groups):

Theorem 1.16. *Assume that one of the following holds:*

(1) *There exists two subsets S_1 and S_2 of S such that $A = S_1 \cup (\cup_{s\in S_2} C(s))$.*
(2) *$S = \{s,t\}$, $m(s,t)$ is even or ∞, and $A = \{s,t,sts\}$ or $A = \{t, sts\}$.*

Then A is nice.

Proof. Assume that (1) or (2) holds. Let $r \in A$ and let $w \in W$ be such that $w^{-1}rw \notin A$. We want to prove that $D_A(rw) = D_A(w)$. By symmetry, we only need to show that $D_A(rw) \subset D_A(w)$. If $r \in S$, then this follows from Corollary 1.2. So we may, and we will, assume that $r \notin S$.

• Assume that (1) holds. Write $A' = \cup_{s\in S_2} C(s)$ and $S' = A \setminus A'$. Then $S' \subset S$, $A = A' \coprod S'$ and A' is stable under conjugacy. Then $r \in A'$ and $w^{-1}rw \in A' \subset A$, which contradicts our hypothesis.

• Assume that (2) holds. If $m(s,t) = 2$, then A is contained in S and therefore is nice by Corollary 1.2. So we may, and we will, assume that $m(s,t) \geq 4$. Since $r \notin S$, we have $r = sts$. Assume that $D_A(rw) \not\subset D_A(w)$. Let

$$\Phi_A = \{\alpha \in \Phi^+ \mid s_\alpha \in A\} \subset \{\alpha_s, \alpha_t, s(\alpha_t)\}.$$

There exists $\alpha \in \Phi_A$ such that $rw(\alpha) \in \Phi^-$ and $w(\alpha) \in \Phi^+$. So $w(\alpha) \in N(r) = \{\alpha_s, s(\alpha_t), st(\alpha_s)\}$. Since $w^{-1}rw \notin A$, we have that $ws_\alpha w^{-1} \neq r$, so $w(\alpha) \neq s(\alpha_t)$. So $w(\alpha) = \alpha_s$ or $st(\alpha_s)$. But the roots α_s and α_t lie in different W-orbits. So $\alpha = \alpha_s$ and $w(\alpha_s) \in \{\alpha_s, st(\alpha_s)\}$. If W is infinite, this gives that $w \in \{1, st\}$. This contradicts the fact that $w^{-1}rw \notin A$. If W is finite, this gives that $w \in \{1, st, w_0 s, stsw_0\}$. But again, this contradicts the fact that $w^{-1}rw \notin A$. $\square$

Example 1.17 - Assume here that W is of type F_4, that $S = \{s_1, s_2, s_3, s_4\}$ and that $A = C(s_1) \cup S$. Then, using `GAP`, one can see that $\mathrm{rank}_{\mathbb{Z}} \mathcal{D}_A(W) = 300$ and $\mathrm{rank}_{\mathbb{Z}} \Sigma_A(W) = 149$. Moreover, $\Sigma_A(W)$ is not a subalgebra of $\mathcal{D}_A(W)$. $\square$

Some computations with `GAP` suggest that the following question has a positive answer:

Question. *Let A be a nice subset of T containing S. Is it true that A is one of the subsets mentioned in Theorem 1.16?*

2. The dihedral groups

The aim of this section is to study the unitary subalgebras $\mathcal{D}_A(W)$ constructed in Theorems A and B whenever W is finite and dihedral.

Hypothesis: *From now on, and until the end of this paper, we assume that $S = \{s,t\}$ with $s \neq t$ and that $m(s,t) = 2m$, with $2 \le m < \infty$.*

Note that $w_0 = (st)^m$ is central. We denote by $\mathcal{P}_0(A)$ the set of subsets I of A such that $W_I \cap A = I$. With this notation, we have

$$\Sigma_A(W) = \sum_{I \in \mathcal{P}_0(A)} \mathbb{Z} x_I.$$

We now define an equivalence relation $\equiv$ on $\mathcal{P}_0(A)$: if I, $J \in \mathcal{P}_0(A)$, we write $I \equiv J$ if W_I and W_J are conjugate in W. We set

$$\Sigma_A^{(1)}(W) = \sum_{\substack{I,J \in \mathcal{P}_0(A) \\ I \equiv J}} \mathbb{Z}(x_I - x_J).$$

In what follows, we will also need some facts on the character table of W. Let us recall here the construction of $\operatorname{Irr} W$. First, let H be the subgroup of W generated by st. It is normal in W, of order $2m$ (in other words, of index 2). We choose the primitive $(2m)$-th root of unity $\zeta \in \mathbb{C}$ of argument $\frac{\pi}{m}$. If $i \in \mathbb{Z}$, we denote by $\xi_i : H \to \mathbb{C}^\times$ the unique linear character such that $\xi_i(st) = \zeta^i$. Then $\operatorname{Irr} H = \{\xi_i \mid 0 \le i \le 2m-1\}$. Now, let

$$\chi_i = \operatorname{Ind}_H^W \xi_i.$$

Then $\chi_i = \chi_{2m-i}$ and, if $1 \le i \le m-1$, $\chi_i \in \operatorname{Irr} W$. Also, χ_i has values in $\mathbb{R}$. More precisely, for $1 \le i \le m-1$ and $j \in \mathbb{Z}$

$$\chi_i\big((ts)^j\big) = \zeta^j + \zeta^{-j} = 2\cos\left(\frac{ij\pi}{m}\right) \quad \text{and} \quad \chi_i\big(s(ts)^j\big) = 0.$$

Let 1 denote the trivial character of W, let ε denote the sign character and let $\gamma : W \to \{1,-1\}$ be the unique linear character such that $\gamma(s) = -\gamma(t) = 1$. Then

$$\textbf{(2.1)} \qquad \operatorname{Irr} W = \{1, \varepsilon, \gamma, \varepsilon\gamma\} \cup \{\chi_i \mid 1 \le i \le m-1\}.$$

In particular, $|\operatorname{Irr} W| = m+3$.

2.1. **The subset $A = \{s,t,sts\}$.** From now on, we assume that $A = \{s,t,sts\}$. We set $\bar{s} = A \setminus \{s\} = \{t, sts\}$ and $\bar{t} = A \setminus \{t\} = \{s, sts\}$. It is easy to see that

$$\mathcal{P}_{\mathrm{ad}}(A) = \{\varnothing, \{s\}, \{t\}, \bar{s}, \bar{t}, A\}.$$

For simplification, we will denote by d_I the element d_I^A of $\mathbb{Z}W$ (for $I \in \mathcal{P}_{\mathrm{ad}}(W)$) and we set $d_s = d_{\{s\}}$ and $d_t = d_{\{t\}}$. We have

$$\begin{array}{rclrcl}
d_\varnothing &=& 1, & d_{\bar{s}} &=& w_0 s, \\
d_s &=& s, & d_A &=& w_0, \\
d_t &=& \displaystyle\sum_{i=1}^{m-1} \Big((st)^i + (ts)^{i-1} t\Big), & d_{\bar{t}} &=& \displaystyle\sum_{i=1}^{m-1} \Big((st)^i s + (ts)^i\Big).
\end{array}$$

The multiplication table of $\mathcal{D}_A(W)$ is given by

	1	d_s	$d_{\bar{s}}$	d_A	d_t	$d_{\bar{t}}$
1	1	d_s	$d_{\bar{s}}$	d_A	d_t	$d_{\bar{t}}$
d_s	d_s	1	d_A	$d_{\bar{s}}$	d_t	$d_{\bar{t}}$
$d_{\bar{s}}$	$d_{\bar{s}}$	d_A	1	d_s	$d_{\bar{t}}$	d_t
d_A	d_A	$d_{\bar{s}}$	d_s	1	$d_{\bar{t}}$	d_t
d_t	d_t	$d_{\bar{t}}$	d_t	$d_{\bar{t}}$	z_A	z_A
$d_{\bar{t}}$	$d_{\bar{t}}$	d_t	$d_{\bar{t}}$	d_t	z_A	z_A

where $z_A = (m-1)\big(1 + d_A + d_s + d_{\bar{s}}\big) + (m-2)\big(d_t + d_{\bar{t}}\big)$. We now study the sub-$\mathbb{Z}$-module $\Sigma_A(W)$: we will show that it coincides with $\mathcal{D}_A(W)$. First, it is easily seen that

$$\mathcal{P}_0(A) = \{\varnothing,\, \{s\},\, \{t\},\, \{sts\},\, \bar{s},\, A\}$$

and that

$$\begin{array}{rcllllll}
x_A & = & 1 & & & & & \\
x_{\bar{s}} & = & 1 & +d_s & & & & \\
x_{sts} & = & 1 & +d_s & +d_t & & & \\
x_t & = & 1 & +d_s & & +d_{\bar{t}} & & \\
x_s & = & 1 & & +d_t & & +d_{\bar{s}} & \\
x_\varnothing & = & 1 & +d_s & +d_t & +d_{\bar{t}} & +d_{\bar{s}} & +d_A
\end{array}$$

Therefore, $\Sigma_A(W) = \mathcal{D}_A(W) = \oplus_{I \in \mathcal{P}_0(A)} \mathbb{Z} x_I$. So we can define a map $\theta_A : \Sigma_A(W) \to \mathbb{Z}\operatorname{Irr} W$ by $\theta_A(x_I) = \operatorname{Ind}_{W_I}^W 1_{W_I}$.

Proposition 2.2. *Assume that $S = \{s,t\}$ with $s \neq t$, that $m(s,t) = 2m$ with $m \geq 2$ and that $A = \{s, t, sts\}$. Then:*

(a) *$\Sigma_A(W) = \mathcal{D}_A(W)$ is a subalgebra of $\mathbb{Z}W$ of $\mathbb{Z}$-rank 6.*
(b) *θ_A is a morphism of algebras.*
(c) *$\operatorname{Ker}\theta_A = \mathbb{Z}(x_t - x_{sts}) = \Sigma_A^{(1)}(W)$.*
(d) *$\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Ker}\theta_A$ is the radical of $\mathbb{Q} \otimes_{\mathbb{Z}} \Sigma_A(W)$.*
(e) *θ_A is surjective if and only if $m = 2$ that is, if and only if W is of type B_2.*

Proof. (a) has already been proved. For proving the other assertions, we need to compute explicitly the map θ_A. It is given by the following table:

d_I	1	d_s	$d_{\bar{s}}$	d_A	d_t	$d_{\bar{t}}$
$\theta_A(d_I)$	1	$\varepsilon\gamma$	γ	ε	$\displaystyle\sum_{i=1}^{m-1}\chi_i$	$\displaystyle\sum_{i=1}^{m-1}\chi_i$

(c) This shows that $\operatorname{Ker}\theta_A = \mathbb{Z}(d_{\bar{t}} - d_t) = \mathbb{Z}(x_t - x_{sts}) = \Sigma_A^{(1)}(W)$, so (c) holds.

(b) To prove that θ_A is a morphism of algebras, the only difficult point is to prove that $\theta_A(d_t^2) = \theta_A(d_t)^2$. Let ρ denote the regular character of W. Then

$$\theta_A(d_t) = \frac{1}{2}\big(\rho - 1 - \gamma - \varepsilon\gamma - \varepsilon\big).$$

Therefore,

$$\theta_A(d_t)^2 = (m-2)\rho + 1 + \gamma + \varepsilon\gamma + \varepsilon.$$

But, $d_t^2 = z_A = (m-2)x_\varnothing + 1 + d_s + d_{\bar{s}} + d_A$. This shows that $\theta_A(d_t^2) = \theta_A(d_t)^2$.

(d) Let R denote the radical of $\mathbb{Q} \otimes_\mathbb{Z} \Sigma_A(W)$. We only need to prove that $\mathbb{C} \otimes_\mathbb{Q} R = \mathbb{C} \otimes_\mathbb{Z} \Sigma_A(W)$. Since $\mathbb{C}\operatorname{Irr} W$ is a split semisimple commutative algebra, every subalgebra of $\mathbb{C}\operatorname{Irr} W$ is semisimple. So $(\mathbb{C} \otimes_\mathbb{Z} \Sigma_A(W))/(\mathbb{C} \otimes_\mathbb{Z} \operatorname{Ker}\theta_A)$ is a semisimple algebra. This shows that R is contained in $\mathbb{C} \otimes_\mathbb{Z} \operatorname{Ker}\theta_A$. Moreover, since $(x_t - s_{sts})^2 = (d_t - d_{\bar{t}})^2 = 0$, $\operatorname{Ker}\theta_A$ is a nilpotent two-sided ideal of $\Sigma_A(W)$. So $\mathbb{C} \otimes_\mathbb{Z} \operatorname{Ker}\theta_A$ is contained in $\mathbb{C} \otimes_\mathbb{Z} R$. This shows (d).

(e) If $m = 2$, then $\operatorname{Irr} W = \{1, \gamma, \varepsilon\gamma, \varepsilon, \chi_1\} = \theta_A(\{1, d_s, d_{\bar{s}}, d_A, d_t\})$ so θ_A is surjective. Conversely, if θ_A is surjective, then $|\operatorname{Irr} W| = 5$ (by (a) and (c)). Since $|\operatorname{Irr} W| = m+3$, this gives $m = 2$. $\square$

We close this subsection by giving, for $\Sigma_A(W)$, a complete set of orthogonal primitive idempotents, extending to our case those given in [2], and the corresponding irreducible representations:

$$E_\varnothing = \frac{1}{4m} x_\varnothing, \qquad E_s = \frac{1}{2}\left(x_s - \frac{1}{2}x_\varnothing\right), \qquad E_t = \frac{1}{2}\left(x_t - \frac{1}{2}x_\varnothing\right),$$

$$\begin{aligned} E_{\bar{s}} &= \frac{1}{2}\left(x_{\bar{s}} - \frac{1}{2}x_t - \frac{1}{2}x_{sts} + \frac{m-1}{2m}x_\varnothing\right) \\ E_A &= 1 - \frac{1}{2}x_s - \frac{1}{4}x_t + \frac{1}{4}x_{sts} - \frac{1}{2}x_{\bar{s}} + \frac{1}{4}x_\varnothing \end{aligned}$$

For $i \in \{\varnothing, s, t, \bar{s}, A\}$, we denote by P_i the indecomposable projective module $\mathbb{Q}\Sigma_A(W)E_i$. Let $\mathbb{Q}_i^A$ denote the unique simple $\mathbb{Q}\Sigma_A(W)$-module lying in the head of P_i. It is easily seen, using the previous multiplication table, that

$$\dim P_\varnothing = \dim P_t = \dim P_{\bar{s}} = \dim P_A = 1$$

and that

$$\dim P_s = 2$$

(note that $x_{\bar{s}}E_s = -(x_t - x_{sts})$). In other words,

$$P_\varnothing = \mathbb{Q}_\varnothing^A, \quad P_t = \mathbb{Q}_t^A, \quad P_{\bar{s}} = \mathbb{Q}_{\bar{s}}^A, \quad P_A = \mathbb{Q}_A^A$$

and

$$\operatorname{Rad}(P_s) \simeq \mathbb{Q}_t^A.$$

For $w \in W$, we denote by ev_w the morphism of algebras $\mathbb{Z}\operatorname{Irr} W \to \mathbb{Z}$, $\chi \mapsto \chi(w)$ and we set $\operatorname{ev}_w^A = \operatorname{ev}_w \circ \theta_A$. Then the morphism of algebras $\Sigma_A(W) \to \mathbb{Z}$ associated to the simple module $\mathbb{Q}_i^A$ is $\operatorname{ev}_{f(i)}^A$, where

$$f(\varnothing) = 1, \quad f(s) = s, \quad f(t) = t, \quad f(\bar{s}) = (st)^m = w_0 \quad \text{and} \quad f(A) = st.$$

2.2. **The subset $B = \{s\} \cup C(t)$.** Let $B = \{s\} \cup C(t)$ (so that $|B| = m+1$). It is easy to see that $\mathcal{P}_{\mathrm{ad}}(B)$ consists of the sets

$$\begin{array}{ccl} \varnothing &=& D_B(1), \\ B &=& D_B(w_0), \\ \{s\} &=& D_B(s), \\ C(t) &=& D_B(w_0 s), \\ D_B\big((ts)^i\big) &=& D_B\big(s(ts)^i\big), \quad 1 \le i \le m-1, \\ D_B\big((st)^j\big) &=& D_B\big((ts)^{j-1}t\big), \quad 1 \le j \le m-1. \end{array}$$

Therefore $\mathcal{D}_B(W)$ is a subalgebra of $\mathbb{Z}W$ of $\mathbb{Z}$-rank $(2m+2)$.

Using GAP, we can see that, in general, $\Sigma_B(W) \neq \mathcal{D}_B(W)$. First examples are given in the following table

m	2	3	4	5	6	7	8	9	10	11
$\mathbb{Z}$-rank of $\mathcal{D}_B(W)$	6	8	10	12	14	16	18	20	22	24
$\mathbb{Z}$-rank of $\Sigma_B(W)$	6	8	10	10	14	12	18	18	22	16

Remark 2.3 - The linear map $\theta_B : \Sigma_B(W) \to \mathbb{Z}\operatorname{Irr} W$, $x_I \mapsto \operatorname{Ind}_{W_I}^W 1_{W_I}$, is well-defined and surjective if and only if $m \in \{2,3\}$ (recall that $m \geq 2$). Indeed, the image of θ_B can not contain a non-rational character. But all characters of W are rational if and only if W is a Weyl group, that is, if and only if $2m \in \{2,3,4,6\}$.

Moreover, if $m = 2$, then $A = B$ and $\theta_B = \theta_A$ is surjective by Proposition 2.2. If $m = 3$, then this follows from Proposition 2.4 below. □

2.3. **The algebra $\mathcal{D}_B(G_2)$.** From now on, and until the end, we assume that $m = 3$. That is W is of type $I_2(6) = G_2$. For convenience, we keep the same notation as in §2.1. We have

$$\begin{array}{rclrcl}
d_\varnothing & = & 1, & d_1 & = & d^B_{\{t\}} = t + st, \\
d_s & = & s, & d_2 & = & d^B_{\{s,sts\}} = ts + sts, \\
d_{\bar{s}} & = & d^B_{C(t)} = w_0 s, & d_3 & = & d^B_{\{s,sts,tstst\}} = tsts + ststs, \\
d_A & = & d^B_B = w_0, & d_4 & = & d^B_{\{t,tstst\}} = tst + stst.
\end{array}$$

Let us now show that $\Sigma_B(W) = \mathcal{D}_B(W)$. First, it is easily seen that

$$\mathcal{P}_0(B) = \{\varnothing,\ \{s\},\ \{t\},\ \{sts\},\ \bar{s},\ \{tstst\}, \{s, tstst\}, B\}$$

and that

$$\begin{array}{rclllllllll}
x_A = x_B & = & 1 & & & & & & & \\
x_{\bar{s}} & = & 1 & +d_s & & & & & & \\
x_{\{s,tstst\}} & = & 1 & & +d_1 & & & & & \\
x_{tstst} & = & 1 & +d_s & +d_1 & +d_2 & & & & \\
x_t & = & 1 & +d_s & & +d_2 & +d_3 & & & \\
x_{sts} & = & 1 & +d_s & +d_1 & & & +d_4 & & \\
x_s & = & 1 & & +d_1 & & & +d_4 & +d_{\bar{s}} & \\
x_\varnothing & = & 1 & +d_s & +d_1 & +d_2 & +d_3 & +d_4 & +d_{\bar{s}} & +d_A
\end{array}$$

Therefore, $\Sigma_B(W) = \mathcal{D}_B(W) = \oplus_{I \in \mathcal{P}_0(B)} \mathbb{Z} x_I$. So we can define a map θ_B : $\Sigma_B(W) \to \mathbb{Z}\operatorname{Irr} W$ by $\theta_B(x_I) = \operatorname{Ind}_{W_I}^W 1_{W_I}$.

Proposition 2.4. *Assume that $S = \{s,t\}$ with $s \neq t$, that $m(s,t) = 6$ and that $B = \{s, t, sts, tstst\}$. Then:*

(a) *$\Sigma_B(W) = \mathcal{D}_B(W)$ is a subalgebra of $\mathbb{Z}W$ of $\mathbb{Z}$-rank 8.*
(b) *θ_B is a surjective linear map (and not a morphism of algebras).*
(c) *$\operatorname{Ker}\theta_B = \mathbb{Z}(x_{tstst} - x_t) \oplus \mathbb{Z}(x_{tstst} - x_{sts}) = \Sigma_B^{(1)}(W)$.*
(d) *$\operatorname{Irr} W = \theta_B(\{1, d_s, d_{\bar{s}}, d_A, d_1, d_2\})$.*

Proof. The map θ_B is given by the following table:

d_I^B	1	d_s	$d_{\bar{s}}$	d_A	d_1	d_2	d_3	d_4
$\theta_B(d_I^B)$	1	$\varepsilon\gamma$	γ	ε	χ_2	χ_1	χ_2	χ_1

This shows (b) and (d). As $d_1 - d_3 = x_{\{tstst\}} - x_t$ and $d_2 - d_4 = x_{tstst} - x_{sts}$, (c) is proved. Finally, the fact that θ_B is not a morphism of algebras follows from $\theta_B(d_1^2)(w_0) = (1 + \varepsilon\gamma + \chi_1)(w_0) = -2 \neq \theta_B(d_1)^2(w_0) = \chi_2(w_0)^2 = 4$. $\square$

For information, the multiplication table in the basis $(d_I^B)_{I \in \mathcal{P}_{\mathrm{ad}}(B)}$ is given by

	1	d_s	d_1	d_2	d_3	d_4	$d_{\bar{s}}$	d_A
1	1	d_s	d_1	d_2	d_3	d_4	$d_{\bar{s}}$	d_A
d_s	d_s	1	d_1	d_2	d_3	d_4	d_A	$d_{\bar{s}}$
d_1	d_1	d_2	$1 + d_s + d_4$	$1 + d_s + d_3$	$d_2 + d_{\bar{s}} + d_A$	$d_1 + d_{\bar{s}} + d_A$	d_4	d_3
d_2	d_2	d_1	$1 + d_s + d_4$	$1 + d_s + d_3$	$d_2 + d_{\bar{s}} + d_A$	$d_1 + d_{\bar{s}} + d_A$	d_3	d_4
d_3	d_3	d_4	$d_2 + d_{\bar{s}} + d_A$	$d_1 + d_{\bar{s}} + d_A$	$1 + d_s + d_4$	$1 + d_s + d_3$	d_2	d_1
d_4	d_4	d_3	$d_2 + d_{\bar{s}} + d_A$	$d_1 + d_{\bar{s}} + d_A$	$1 + d_s + d_4$	$1 + d_s + d_3$	d_1	d_2
$d_{\bar{s}}$	$d_{\bar{s}}$	d_A	d_3	d_4	d_1	d_2	1	d_s
d_A	d_A	$d_{\bar{s}}$	d_3	d_4	d_1	d_2	d_s	1

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